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Diffraction of Kelvin waves at a sharp bend

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A closed form solution is obtained for the linearized problem of the diffraction of Kelvin waves by a sharp bend on a rotating earth. It is shown that for the frequency ω of the incoming wave less than f, where $\frac{1}{2}f$ is the angular velocity of rotation, the wave is transmitted around the bend without change of amplitude. For $\omega > f$ the amplitude is in general reduced but is unaltered for the special angles $\pi/(2n+1)$. For these special angles the solution is obtained in elementary terms.

1. Introduction

The diffraction of a Kelvin wave at a right-angled bend in a straight coast line which bounds a sheet of water of uniform depth rotating with constant angular velocity about a vertical axis has recently been considered by Buchwald (1968) using a Wiener–Hopf technique.

His method is only directly applicable to the case of a right-angled wedge and it is therefore desirable to consider the wave field due to an incident Kelvin wave for a wedge of arbitrary angle.

In this paper, the solution is obtained using a method employed by Williams (1959) and extended by Faulkner (1965) for problems concerning the diffraction of electromagnetic waves by wedges. The method depends on choosing a suitable contour integral representation for the solution and reducing the boundary-value problem to the solution of a difference equation. If the frequency of the incident wave is ω and the angular velocity of rotation is $\frac{1}{2}f$, then the solution and choice of contour depend on whether ω is less than or greater than f. In both cases the solution may be expressed in terms of the double gamma function.

Despite the fact that some of the analysis is complex it has been found possible to obtain elementary expressions in each case for the transmission coefficient T defined as the ratio of the amplitude of the transmitted wave to that of the incident wave. For $\omega < f$, $T = 1$; that is Kelvin waves are propagated round a corner of any angle without change of amplitude. This agrees with Buchwald's result for the special case of a right-angled wedge. For $\omega > f$, $T \leqslant 1$, where equality occurs at all frequencies for wedge angles $\pi/(2n+1)$ (n a positive integer) and for these angles only. The dependence of T on the wedge angle is shown graphically for several values of f/ω. Buchwald has not calculated T for $\omega > f$ for his special case of a right-angled wedge, but the calculation can be made and the result is found to agree with the general expression.

If for $\omega > f$, $T = 1$, which, as we have remarked, only occurs for wedge angles $\pi/(2n+1)$, then from energy considerations there can be no outgoing cylindrical wave, and it is shown that for these wedge angles the solution does not in fact possess a cylindrical wave contribution. Further, for this case, the solution can be written down in elementary finite terms consisting of the incident and transmitted waves together with $(n-1)$ Kelvin waves and n Poincaré waves which are attenuated within and along the wedge.

It should be noted that the problem of diffraction of an incident plane wave is identical to the problem of electromagnetic scattering treated by Faulkner (1965). This problem has also been treated by Roseau (1967) who seems unaware of Faulkner's work.

2. Formulation of the problem

Consider a plane horizontal sheet of water contained between two vertical planes forming a wedge of angle γ. In the undisturbed state the water has a depth h and the whole system has a uniform rotation $\frac{1}{2}f$ about a vertical axis.

The problem is most conveniently stated in terms of a cylindrical polar system of co-ordinates (r, θ, z) rotating with the wedge, such that the edge of the wedge is the z-axis, the base of the sheet is $z = 0$, and the bounding planes are $\theta = 0$ and $\theta = \gamma$. If, in the disturbed state, the equation of the free surface is $z = h + \bar{\zeta}$, then the linearized long wave equations are,

$$\left.\begin{aligned} \bar{u}_t - f\bar{v} &= -g\bar{\zeta}_r, \\ \bar{v}_t + f\bar{u} &= -\frac{g}{r}\bar{\zeta}_\theta, \\ \text{and} \qquad \frac{h}{r}\left\{\frac{\partial}{\partial r}(r\bar{u}) + \frac{\partial \bar{v}}{\partial \theta}\right\} + \bar{\zeta}_t &= 0, \end{aligned}\right\} \tag{2.1}$$

where $\bar{u}(x,y,t)$, $\bar{v}(x,y,t)$ are the horizontal radial and transverse components of velocity.

Assuming $\bar{\zeta}(x,y,t) = \zeta(x,y)\,e^{i\omega t}$, and similarly for $\bar{u}$ and $\bar{v}$, it is easily deduced that

$$hk^2u = i\omega\zeta_r + \frac{f}{r}\zeta_\theta, \tag{2.2}$$

$$hk^2v = -f\zeta_r + \frac{i\omega}{r}\zeta_\theta, \tag{2.3}$$

and
$$(\nabla^2 + k^2)\zeta = 0, \tag{2.4}$$

where $k^2 = (\omega^2 - f^2)/c_0^2$, and $c_0^2 = gh$.

Our boundary-value problem is therefore to find a solution of (2.4) such that

$$f\frac{\partial \zeta}{\partial r} - \frac{i\omega}{r}\frac{\partial \zeta}{\partial \theta} = 0 \quad \text{for} \quad \theta = 0, \gamma, \tag{2.5}$$

and such that the incident wave is an incoming Kelvin wave along the boundary $\theta = \gamma$. Such a wave of unit amplitude is given by

$$\zeta_0 = \exp\left[ikr\cos(\theta - \gamma - \lambda)\right], \tag{2.6}$$

where $f = ic_0k\sin\lambda$ and $\omega = c_0k\cos\lambda$.

In order to obtain a solution of this boundary-value problem we follow Williams's approach and assume that ζ may be defined by

$$\zeta = \int_C f(\nu, \theta) \exp[-ikr\cos\nu]\,d\nu, \tag{2.7}$$

if the contour C and function $f(\nu, \theta)$ are suitably chosen. It is also necessary at this stage to distinguish between the two cases $\omega > f$ and $\omega < f$.

Case I. $\omega > f$. Here k is real and λ is purely imaginary, say $\lambda = -i\lambda_0$, where λ_0 is real, so that in this case

$$f = c_0 k \sinh\lambda_0 \quad \text{and} \quad \omega = c_0 k \cosh\lambda_0. \tag{2.8}$$

Case II. $\omega < f$. Here k is imaginary, say $k = -iK\,(K > 0)$, and λ is of the form $\lambda = \frac{1}{2}\pi - i\lambda_1$, where λ_1 is real. Hence in this case

$$f = c_0 K\cosh\lambda_1 = ic_0 k\cosh\lambda_1 \quad \text{and} \quad \omega = c_0 K\sinh\lambda_1 = ic_0 k\sinh\lambda_1. \tag{2.9}$$

In the case $\omega > f$ the choice of contour is identical to that of Williams and Faulkner, and the problem differs from that considered by Faulkner only in the nature of the incident wave.

The case $\omega < f$, however, requires a different choice of contour and this will be discussed in due course.

3. Solution for the case $\omega > f$

We look for a solution of the form (2.7) in which k is real, where $f(\nu, \theta)$ is a continuous, analytic function of ν and C is a contour which lies entirely in the strip $-\frac{1}{2}\pi < \mathscr{R}\nu < \frac{3}{2}\pi$, with end-points at $-i\infty - \frac{1}{2}\pi$ and $-i\infty + \frac{3}{2}\pi$, and such that it lies below all singularities of $f(\nu, \theta)$. In order to ensure the uniform convergence of the integral and the integrals obtained by replacing $f(\nu, \theta)$ by its first or second derivative with respect to θ, it is necessary to make further justifiable assumptions concerning the behaviour of $f(\nu, \theta)$, the details being given in Williams (1959).

It is then found (1959) that (2.7) is a solution of (2.4) if

$$f(\nu, \theta) = g_1(\nu + \theta) + g_2(\nu - \theta), \tag{3.1}$$

where g_1 and g_2 are arbitrary functions.

If we write

$$J_{\pm} = \int_C g(\nu \pm \theta) \exp[-ikr\cos\nu]\,d\nu,$$

then

$$\frac{\partial J_{\pm}}{\partial r} = ik\int_C \cos\nu\, g(\nu \pm \theta) \exp[-ikr\cos\nu]\,d\nu, \tag{3.2}$$

whilst an integration by parts gives

$$\frac{\partial J_{\pm}}{\partial \theta} = \mp\, ikr\int_C \sin\nu\, g(\nu \pm \theta) \exp[-ikr\cos\nu]\,d\nu. \tag{3.3}$$

It therefore follows that the boundary conditions on $\theta = 0, \gamma$ are satisfied if

$$(\omega \sin \nu + if \cos \nu) g_1(\nu) = (\omega \sin \nu - if \cos \nu) g_2(\nu),$$

and
$$(\omega \sin \nu + if \cos \nu) g_1(\nu + \gamma) = (\omega \sin \nu - if \cos \nu) g_2(\nu - \gamma),$$

or equivalently
$$\sin(\nu - \lambda) g_1(\nu) = \sin(\nu + \lambda) g_2(\nu), \tag{3.4}$$

and
$$\sin(\nu - \lambda) g_1(\nu + \gamma) = \sin(\nu + \lambda) g_2(\nu - \gamma). \tag{3.5}$$

From (3.4) and (3.5) we find that a solution of (2.4) satisfying the boundary conditions (2.5) is

$$\zeta = \int_C \{G(\nu + \theta) + K(\nu - \theta, \lambda) G(\nu - \theta)\} \exp[-ikr \cos \nu] \, d\nu, \tag{3.6}$$

where
$$G(\nu) = G(\nu + 2\gamma) K(\nu + \gamma, \lambda) K(\nu, -\lambda), \tag{3.7}$$

and
$$K(\nu, \lambda) = \frac{\sin(\nu - \lambda)}{\sin(\nu + \lambda)}. \tag{3.8}$$

The problem is thus reduced to finding that solution of the difference equation (3.7) which gives the correct behaviour for ζ at infinity. That is, for large values of r, the solution must consist of the incident Kelvin wave, a transmitted Kelvin wave and an outgoing cylindrical wave.

The normal saddle-point method shows that the saddle point at $\nu = 0$ gives an outgoing cylindrical wave, whilst the saddle point at $\nu = \pi$ gives an incoming cylindrical wave. Using the argument in Williams (1959), the term arising from $\nu = \pi$ vanishes if $G(\nu)$ satisfies the subsidiary condition

$$G(\nu) \sin(\nu - \lambda) = -G(2\pi - \nu) \sin(\nu + \lambda). \tag{3.9}$$

Again using the argument in Williams (1959), the condition that ζ should be finite for large r implies that the integrand of (3.6) must have no poles in the strip $\mathscr{I} \nu < 0$, $0 < \mathscr{R}\nu < \pi$. Finally, there must be poles on $\mathscr{I}\nu = -\lambda_0$ to provide the Kelvin waves and the integrand of (3.6) must satisfy the convergence requirements referred to above.

Now the most general solution of (3.7) which satisfies (3.9) is $G(\nu) = H_0(\nu) H(\nu)$, where $H(\nu)$ is a particular solution and $H_0(\nu)$ is such that

$$H_0(\nu) = H_0(\nu + 2\gamma), \tag{3.10}$$

and
$$H_0(\nu) = H_0(2\pi - \nu). \tag{3.11}$$

A particular solution of (3.7) may be expressed, as in Faulkner (1965) in terms of the solution $F(\nu)$ of the subsidiary equation

$$\sin(\nu + \lambda) F(\nu) = \sin(\nu - \lambda) F(\nu + 2\gamma). \tag{3.12}$$

Such a solution is clearly

$$H_1(\nu) = F(\nu + \gamma)/F(\nu). \tag{3.13}$$

We now put $\nu = 2\gamma\eta$, $\lambda = 2\gamma\mu$ and $F(\nu) = F(2\gamma\eta) = I(\eta)$. Equation (3.12) then becomes

$$\sin[2\gamma(\eta + \mu)] I(\eta) = \sin[2\gamma(\eta - \mu)] I(\eta + 1). \tag{3.14}$$

Consider the function $I(\eta)$ defined by

$$I(\eta) = \frac{M[1+\tau-\eta-\mu,\tau]\,M[\eta-\mu,\tau]}{M[1+\tau-\eta+\mu,\tau]\,M[\eta+\mu,\tau]}, \tag{3.15}$$

where $M(v,\delta)$ is Barnes's (1899) double gamma function which satisfies the difference relation

$$M(v+1,\delta) = \Gamma(v/\delta)\,M(v,\delta), \tag{3.16}$$

and $2\gamma\tau = \pi$.

From (3.15) and (3.16) we obtain

$$\frac{I(\eta)}{I(\eta+1)} = \frac{\Gamma([\eta+\mu]/\tau)\,\Gamma(1-[\eta+\mu]/\tau)}{\Gamma([\eta-\mu]/\tau)\,\Gamma(1-[\eta-\mu]/\tau)} = \frac{\sin 2\gamma(\eta-\mu)}{\sin 2\gamma(\eta+\mu)}. \tag{3.17}$$

It therefore follows that $I(\eta)$ defined by (3.15) is a solution of (3.14). Inspection of (3.15) shows that

$$I(\eta) = I(1+\tau-\eta), \tag{3.18}$$

and by using Barnes's formula

$$M(v+\tau,\tau) = (2\pi)^{\frac{1}{2}(\tau-1)}\tau^{\frac{1}{2}-v}\,\Gamma(v)\,M(v,\tau), \tag{3.19}$$

we obtain

$$I(\eta) = \frac{\sin\pi(\eta-\mu)}{\sin\pi(\eta+\mu)}\,I(\eta+\tau). \tag{3.20}$$

Returning to our original variables we therefore have that the function $F(\nu)$ defined by (3.15) satisfies (3.12) and the relationships

$$F(\nu) = F(2\gamma+\pi-\nu), \tag{3.21}$$

and

$$F(\nu) = \frac{\sin\tau(\nu-\lambda)}{\sin\tau(\nu+\lambda)}\,F(\nu+\pi). \tag{3.22}$$

From equations (3.12), (3.21) and (3.22) we may show that

$$\frac{H_1(\nu)}{H_1(2\pi-\nu)} = -\frac{\sin(\nu+\lambda)}{\sin(\nu-\lambda)}\,\frac{\tan\tau\{\lambda+(2\pi-\nu)-\pi\}}{\tan\tau\{\lambda+\nu-\pi\}}. \tag{3.23}$$

Hence if we put

$$H(\nu) = H_1(\nu)\tan\tau(\lambda+\nu-\pi), \tag{3.24}$$

then $H(\nu)$ satisfies (3.9). Further, since $H_1(\nu)$ is a solution of (3.7) and

$$\tan\tau(\lambda+\nu-\pi)$$

is a solution of (3.10) it follows that $H(\nu)$ is also a solution of (3.7).

Hence $H(\nu)$ is a particular solution of (3.7) which satisfies (3.9).

It now remains to choose $H_0(\nu)$ so that the integrand of (3.6), which in view of (3.9) may be written in the form

$$\zeta = \int_C \{G(\nu+\theta)-G(2\pi+\theta-\nu)\}\exp[-ikr\cos\nu]\,d\nu, \tag{3.25}$$

has no poles in $\mathscr{I}(\nu) = 0$, $0 < \mathscr{R}(\nu) < \pi$, and such that the remaining conditions at infinity are satisfied.

In considering the behaviour of ζ for large values of r it is necessary to deform C into the saddle-point curve illustrated in Williams (1959). The integral over the saddle-point curve gives the outgoing cylindrical wave, whilst the Kelvin waves are given by the residues at those poles on $\mathscr{I}\nu = -\lambda_0$ which are captured in the deformation. A wave will be outgoing if the pole lies in $\mathscr{I}\nu < 0$, $-\frac{1}{2}\pi < \mathscr{R}\nu < 0$, and incoming if the pole lies in $\mathscr{I}\nu < 0$, $\pi < \mathscr{R}\nu < \frac{3}{2}\pi$. For those values of θ for which a pole lies within either of these strips or the strip $\mathscr{I}\nu > 0$, $0 < \mathscr{R}\nu < \pi$, the corresponding wave is exponentially damped. The wave is purely trigonometric and incoming if it lies on $\mathscr{R}\nu = \pi$, and purely trigonometric and outgoing if it lies on $\mathscr{R}\nu = 0$. Hence the incident wave (2.6) corresponds to a pole at

$$\nu = \pi - \theta + \gamma + \lambda,$$

which lies in $\mathscr{I}\,\nu < 0$, $\pi < \mathscr{R}\nu < \frac{3}{2}$ for $0 < \theta < \gamma$ and on $\mathscr{R}\nu = \pi$ for $\theta = \gamma$.

It is now easily verified that all the remaining conditions are satisfied with $H_0(\nu) =$ constant.

For, since the zeros of $M(z, \delta)$ are at $z = -(m\delta + n)$, $m, n = 0, 1, 2 \ldots$, it follows that the poles of $H(\nu + \theta)$ are at

$$\nu = -(m-1)\pi - \lambda - \theta - (2n+1)\gamma, \tag{3.26a}$$

$$\nu = (m+2)\pi - \lambda - \theta + 2(n+1)\gamma, \tag{3.26b}$$

$$\nu = (m+1)\pi + \lambda - \theta + (2n+1)\gamma \tag{3.26c}$$

and

$$\nu = -m\pi + \lambda - \theta - 2n\gamma. \tag{3.26d}$$

Similarly, those of $H(2\pi + \theta - \nu)$ are at

$$\nu = -(m-1)\pi - \lambda + \theta - (2n+1)\gamma, \tag{3.27a}$$

$$\nu = (m+2)\pi - \lambda + \theta + 2n\gamma, \tag{3.27b}$$

$$\nu = (m+1)\pi + \lambda + \theta + (2n+1)\gamma \tag{3.27c}$$

and

$$\nu = -m\pi + \lambda + \theta - 2(n+1)\gamma. \tag{3.27d}$$

Bearing in mind that in this case $\lambda = -i\lambda_0$ $(\lambda_0 > 0)$ we see that there are no poles in the strip $\mathscr{I}\nu < 0$, $0 < \mathscr{R}\nu < \pi$. Also, any pole captured in the upper half plane will give a damped wave, so that poles of the sets (a) and (b) may be ignored in discussing conditions at infinity, as may any pole in the lower half plane which does not lie on $\mathscr{R}\nu = 0$ or $\mathscr{R}\nu = \pi$. The only poles which give a finite contribution at infinity are (3.26c) with $m = n = 0$ and $\theta = \gamma$, and (3.26d) with $m = n = 0$ and $\theta = 0$. The first clearly gives the incident incoming wave along $\theta = \gamma$, and the second is the transmitted Kelvin wave along $\theta = 0$.

It therefore follows that $H_0(\nu)$ is an analytic function having no poles and bounded at infinity, i.e. $H_0(\nu) = A$, where A is a constant.

If the incident wave (2.6) is of unit amplitude then A is determined from the equation

$$2\pi i R_1 = 1, \tag{3.28}$$

where R_1 is the residue at $\nu = \pi + \lambda + \gamma - \theta$ of

$$\{G(\nu + \theta) - G(2\pi + \theta - \nu)\}.$$

Simple manipulation shows that

$$R_1 = A \lim_{\omega \to 0} \omega H(\omega + \pi + \lambda + \gamma). \tag{3.29}$$

The complete solution of the boundary-value problem for the case $\omega > f$ is thus given by

$$\zeta = A \int_C \{H(\nu + \theta) - H(2\pi + \theta - \nu)\} \exp[-ikr\cos\nu]\, d\nu, \tag{3.30}$$

where A is given by (3.28) and $H(\nu)$ is given by (3.24), (3.13) and (3.15).

4. Solution for the case $\omega < f$

We again assume there is a solution of the form (2.7), or equivalently, since in this case $k = -iK$, of the form

$$\zeta = \int_C f(\nu, \theta) \exp[-Kr\cos\nu]\, d\nu. \tag{4.1}$$

It will be assumed, as in §3, that apart from poles $f(\nu, \theta)$ is a continuous, analytic function of ν and is such that all integrals occurring are uniformly convergent. The contour C is chosen to be the line $\mathscr{R}\nu = -\frac{1}{2}\pi$, with end-points $-\frac{1}{2}\pi - i\infty$ and $-\frac{1}{2}\pi + i\infty$, indented at $\nu = -\frac{1}{2}\pi \pm i\lambda_1$ so that the two points lie to the right of the contour. It will be seen later that the indentations are necessary in order that poles of $f(\nu, \theta)$ do not cross the contour for certain values of θ. If this were to happen then ζ would be discontinuous and this is avoided by indenting C.

It then follows from §3, replacing k by $-iK$, that a solution of

$$(\nabla^2 - K^2)\zeta = 0 \tag{4.2}$$

satisfying the boundary conditions (2.5) is

$$\zeta = \int_C \{G(\nu + \theta) + K(\nu - \theta, \lambda)\, G(\nu - \theta)\} \exp[-Kr\cos\nu]\, d\nu, \tag{4.3}$$

where $G(\nu)$ satisfies the same difference equation (3.7) as before and $K(\nu, \lambda)$ is given by (3.8).

The problem is thus reduced to finding that solution of the difference equation which gives the correct behaviour for ζ at infinity. Applying the saddle-point method to the contour integral (4.3) the relevant saddle point is at $\nu = 0$ and the corresponding steepest descent curve is the imaginary axis. In deforming C into the imaginary axis certain poles of the integrand will be captured and ζ will be given by the integral over the steepest descent curve together with terms arising from the captured poles. For large values of r the integral over the steepest descent curve is of the form $Ar^{-\frac{1}{2}} \exp(-Kr)$ so that in this case there is no cylindrical wave and the subsidiary condition (3.9) is not required. The residues at the poles captured in the deformation which lie in $-\frac{1}{2}\pi < \mathscr{R}\nu < 0$ produce terms which are exponentially damped. Purely trigonometric waves will be given by capturing poles at $\nu = -\frac{1}{2}\pi \pm i\lambda_1$, that at $\nu = -\frac{1}{2}\pi + i\lambda_1$ being outgoing and that at $\nu = -\frac{1}{2}\pi - i\lambda_1$ being incoming. The incident Kelvin wave (2.6) must therefore correspond to a pole at $\nu = \lambda + \gamma - \theta - \pi$, and the transmitted wave to a pole at $\nu = \theta - \lambda$. These poles, in view of the indentations, will lie always to the right of C and will be captured in the deformation.

Our object is therefore to construct a solution of (3.7) for which the integrand of (4.3) has the correct poles. Since we have already shown that $H_1(\nu)$, as given by (3.13), is a particular solution of (3.7) the most general solution is

$$G(\nu) = H_1(\nu) H_0(\nu), \tag{4.4}$$

where $H_0(\nu)$ is bounded at the end-points of C and satisfies (3.10).

The problem is thus reduced to finding $H_0(\nu)$ such that any inadmissible poles of the integrand are cancelled, and, if necessary, poles corresponding to the incident and transmitted waves are introduced. Now it follows from (3.12) and (3.13) that

$$H_1(\nu) H_1(\nu + \gamma) = \frac{\sin(\nu + \lambda)}{\sin(\nu - \lambda)}, \tag{4.5}$$

and from (3.13) and (3.21) that

$$H_1(\pi + \nu) H_1(\gamma - \nu) = 1. \tag{4.6}$$

Hence (4.3) may also be written in the form

$$\zeta = \int_C \{H_1(\nu + \theta) H_0(\nu + \theta) + H_0(\nu - \theta) H_1(\pi + \theta - \nu)\} \exp[-Kr\cos\nu]\, d\nu. \tag{4.7}$$

Since the zeros of $M(z, \delta)$ are at $z = -(m\delta + n)$, $m, n = 0, 1, 2, \ldots$, it follows that the poles of $H_1(\nu + \theta)$ are at

$$\nu = -(m + \tfrac{1}{2})\pi + i\lambda_1 - (2n+1)\gamma - \theta, \tag{4.8a}$$

$$\nu = (m + \tfrac{3}{2})\pi - i\lambda_1 + (2n+1)\gamma - \theta, \tag{4.8b}$$

$$\nu = (m + \tfrac{1}{2})\pi + i\lambda_1 + 2(n+1)\gamma - \theta \tag{4.8c}$$

and

$$\nu = -(m - \tfrac{1}{2})\pi - i\lambda_1 - 2n\gamma - \theta. \tag{4.8d}$$

Similarly, those of $H_1(\pi + \theta - \nu)$ are at

$$\nu = -(m + \tfrac{1}{2})\pi + i\lambda_1 - (2n+1)\gamma + \theta, \tag{4.9a}$$

$$\nu = (m + \tfrac{3}{2})\pi - i\lambda_1 + (2n+1)\gamma + \theta, \tag{4.9b}$$

$$\nu = (m + \tfrac{1}{2})\pi + i\lambda_1 + 2n\gamma + \theta \tag{4.9c}$$

and

$$\nu = -(m - \tfrac{1}{2})\pi - i\lambda_1 - 2(n+1)\gamma + \theta. \tag{4.9d}$$

Of these the inadmissible poles are

$$\nu = \tfrac{1}{2}\pi - i\lambda_1 - 2n\gamma - \theta, \tag{4.10}$$

$$\nu = \tfrac{1}{2}\pi - i\lambda_1 - 2(n+1)\gamma + \theta, \tag{4.11}$$

$$\nu = -\tfrac{1}{2}\pi - i\lambda_1 - \theta \tag{4.12}$$

and

$$\nu = -\tfrac{1}{2}\pi + i\lambda_1 - \gamma + \theta. \tag{4.13}$$

The poles (4.10) and (4.11) are cancelled if $H_0(\nu)$ has a factor $\tan\tau(\nu-\lambda)$, the pole (4.12) by a factor $\tan\tau(\nu-\lambda+\pi)$ and the pole (4.13) by a factor

$$\tan\tau(\nu+\lambda+\gamma) = \cot\tau(\nu+\lambda).$$

It is now easily verified that these same factors introduce the required incident and transmitted waves and that the solution is given by

$$H_0(\nu) = A'\tan\tau(\nu-\lambda)\tan\tau(\nu-\lambda+\pi)\cot\tau(\nu+\lambda), \tag{4.14}$$

where A' is a constant. Clearly $H_0(\nu)$ is bounded at the end points of C.

With this choice of $H_0(\nu)$ a straightforward calculation taking account of the cancellation of poles and zeros shows that the poles of $H_0(\nu+\theta)H_1(\nu+\theta)$ are at

$$\nu = -(m+\tfrac{3}{2})\pi+i\lambda_1-(2n+1)\gamma-\theta, \tag{4.15a}$$

$$\nu = (m-\tfrac{1}{2})\pi-i\lambda_1+(2n+1)\gamma-\theta, \tag{4.15b}$$

$$\nu = (m-\tfrac{1}{2})\pi+i\lambda_1+2(n+1)\gamma-\theta \tag{4.15c}$$

and

$$\nu = -(m+\tfrac{3}{2})\pi-i\lambda_1-2n\gamma-\theta. \tag{4.15d}$$

Similarly, those of $H_0(\nu-\theta)H_1(\pi+\theta-\nu)$ are at

$$\nu = (m-\tfrac{1}{2})\pi+i\lambda_1+2n\gamma+\theta, \tag{4.16a}$$

$$\nu = -(m+\tfrac{3}{2})\pi-i\lambda_1-2(n+1)\gamma+\theta, \tag{4.16b}$$

$$\nu = -(m+\tfrac{3}{2})\pi+i\lambda_1-(2n+1)\gamma+\theta \tag{4.16c}$$

and

$$\nu = (m-\tfrac{1}{2})\pi-i\lambda_1+(2n+1)\gamma+\theta. \tag{4.16d}$$

The only poles giving a finite contribution at infinity are (4.15*b*) with $m = n = 0$ and $\theta = \gamma$, and (4.16*a*) with $m = n = 0$ and $\theta = 0$. The first clearly gives the incident wave along $\theta = \gamma$, and the second is the transmitted wave along $\theta = 0$. Any other pole either lies to the left of $\mathscr{R}\nu = -\tfrac{3}{2}\pi$ and may be ignored in discussing conditions at infinity or gives rise to a damped wave if it is captured in the deformation.

The required solution is therefore given by (4.3) with

$$G(\nu) = A'H_1(\nu)\tan\tau(\nu-\lambda)\tan\tau(\nu-\lambda+\pi)\cot\tau(\nu+\lambda), \tag{4.17}$$

or, since from (3.13) and (3.22)

$$H_1(\pi+\nu) = H_1(\nu)\tan\tau(\nu-\lambda)\cot\tau(\nu+\lambda), \tag{4.18}$$

by

$$G(\nu) = A'H_1(\pi+\nu)\tan\tau(\nu-\lambda+\pi). \tag{4.19}$$

In this case if the incident wave (2.6) is of unit amplitude then A' is determined from the equation

$$2\pi i R_1' = -1, \tag{4.20}$$

where R_1' is the residue at $\nu = \lambda+\gamma-\theta-\pi$ of $\{G(\nu+\theta)+K(\nu-\theta,\lambda)G(\nu-\theta)\}$ that is

$$R_1' = \lim_{\omega\to 0}\{\omega G(\omega+\lambda+\gamma-\pi)\}. \tag{4.21}$$

5. The asymptotic solution

In the derivation of our solutions we have already discussed in some detail the form of the solution for large values of r. In both cases, for sufficiently large values of r, the solution consists essentially of the incident trigonometric wave along the wall $\theta = \gamma$ and a transmitted trigonometric wave along $\theta = 0$. Since the ratio of the amplitudes of the transmitted and incident waves for large values of r is the quantity of chief physical significance we devote this section to its calculation.

We consider first the case $\omega > f$. Let the transmitted wave be

$$B \exp[-ikr\cos(\theta-\lambda)]$$

when the incident wave is given by (2.6), then

$$B = 2\pi i R_2, \tag{5.1}$$

where R_2 is the residue at $\nu = \lambda - \theta$ of $\{G(\nu+\theta) - G(2\pi+\theta-\nu)\}$.

Now

$$R_2 = A \lim_{\omega\to 0} \omega H(\omega+\lambda),$$

so that from (3.28) and (3.29) it follows that

$$\begin{aligned} B &= \lim_{\omega\to 0} \frac{H(\omega+\lambda)}{H(\omega+\pi+\lambda+\gamma)} \\ &= \lim_{\omega\to 0} \frac{H_1(\omega+\lambda)}{H_1(\omega+\pi+\lambda+\gamma)} \frac{\tan\tau(2\lambda-\pi)}{\tan\tau(2\lambda+\gamma)} \\ &= \tan\tau(\pi-2\lambda)\tan 2\tau\lambda \lim_{\eta\to 0} \frac{I(\eta+\mu+\frac{1}{2})\, I(\eta+\tau+\mu+\frac{1}{2})}{I(\eta+\mu)\, I(\eta+\tau+\mu)}. \end{aligned}$$

Using (3.15), this becomes, after some reduction,

$$B = -\tan\tau(\pi-2\lambda)\tan 2\tau\lambda \times \frac{M[\frac{1}{2}+\tau-2\mu,\tau]\,M[2\mu,\tau]\,M[\frac{1}{2}-2\mu,\tau]\,M[1+\tau+2\mu,\tau]}{M[\frac{1}{2}+\tau+2\mu,\tau]\,M[-2\mu,\tau]\,M[\frac{1}{2}+2\mu,\tau]\,M[1+\tau-2\mu,\tau]}.$$

Since in the part which involves the double-gamma function the numerator and denominator are conjugate functions, we have

$$|B| = |\tan\tau(\pi-2\lambda)\tan 2\lambda\tau|. \tag{5.2}$$

In view of the linearity of the problem this is clearly the required transmission coefficient T.

A similar, but rather more complicated, calculation can be made for $\omega < f$. In this case if the transmitted wave is $B' \exp[-Kr\cos(\theta-\lambda)]$ then

$$B' = -2\pi i R_2', \tag{5.3}$$

where R_2' is the residue at $\nu = \theta - \lambda$ of $\{G(\nu+\theta) + K(\nu-\theta,\lambda)\, G(\nu-\theta)\}$. Now

$$R_2' = \lim_{\omega\to 0} \{\sin(\omega - 2\lambda)\, G(\omega-\lambda)\},$$

so that from (4.20) and (4.21) it follows that

$$B' = \lim_{\omega\to 0}\frac{\sin(\omega-2\lambda)\,G(\omega-\lambda)}{\omega G(\omega+\lambda+\gamma-\pi)}$$

$$= \lim_{\omega\to 0}\frac{\sin(\omega-2\lambda)\,H_1(\omega-\lambda+\pi)\tan\tau(\omega-2\lambda+\pi)}{\omega H_1(\omega+\lambda+\gamma)\tan\tau(\omega+\gamma)}$$

$$= \tau\sin 2\lambda\tan\tau(\pi-2\lambda)\,\frac{H_1(\pi-\lambda)}{H_1(\lambda+\gamma)}.$$

Hence, using (4.6),

$$|B'| = \tau\sinh 2\lambda_1\tanh(\pi\lambda_1/\gamma)\,|H_1^2(\pi-\lambda)|. \tag{5.4}$$

Now $$H_1(\pi-\lambda) = \frac{I(\mu+\frac{1}{2})}{I(\mu+1)}$$

$$= \frac{M[\frac{1}{2}+\tau-2\mu,\tau]\,M[\frac{1}{2},\tau]\,M[1+2\mu,\tau]\,M[\tau,\tau]}{M[\frac{1}{2}+2\mu,\tau]\,M[\frac{1}{2}+\tau,\tau]\,M[\tau-2\mu,\tau]\,M[1,\tau]},$$

or making use of Barnes's formulae (3.16) and (3.19),

$$H_1(\pi-\lambda) = \frac{\Gamma(1-\tau+2\mu)\,\Gamma(1-[2\mu/\tau])\,M[\frac{1}{2}+\tau-2\mu,\tau]\,M[1-\tau+2\mu,\tau]}{\tau\Gamma(\frac{1}{2}-\tau+2\mu)\,\Gamma(\frac{1}{2})\,M[\frac{1}{2}-\tau+2\mu,\tau]\,M[1+\tau-2\mu,\tau]}. \tag{5.5}$$

Since $\tau-2\mu = i\lambda_1/\gamma$, it follows, as before, that the part involving the double-gamma function in (5.5) is of unit modulus and

$$|H_1(\pi-\lambda)| = \left|\frac{\Gamma(1-[i\lambda_1/\gamma])\,\Gamma(i\lambda_1/\gamma\tau)}{\tau\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-[i\lambda_1/\gamma])}\right|$$

$$= \left|\frac{\Gamma(\frac{1}{2})\,\Gamma(i\lambda_1/\gamma\tau)}{\tau\sin(i\pi\lambda_1/\gamma)\,\Gamma(i\lambda_1/\gamma)\,\Gamma(\frac{1}{2}-[i\lambda_1/\gamma])}\right|.$$

Hence

$$|H_1(\pi-\lambda)|^2 = \frac{\pi}{\tau^2\sinh^2(\pi\lambda_1/\gamma)}\,\frac{(\lambda_1/\gamma)\sinh(\pi\lambda_1/\gamma)}{(\lambda_1/\gamma\tau)\sinh(\pi\lambda_1/\gamma\tau)}\left|\frac{1}{\Gamma(\frac{1}{2}-[i\lambda_1/\gamma])\,\Gamma(\frac{1}{2}+[i\lambda_1/\gamma])}\right|$$

$$= \frac{\pi}{\tau}\operatorname{cosech}\frac{\pi\lambda_1}{\gamma}\operatorname{cosech}2\lambda_1\left|\frac{\sin\pi(\frac{1}{2}-[i\lambda_1/\gamma])}{\pi}\right|$$

$$= \left(\tau\sinh 2\lambda_1\tanh\frac{\pi\lambda_1}{\gamma}\right)^{-1}.$$

Thus from (5.4) $$|B'| = 1 = T. \tag{5.6}$$

It follows that for $\omega < f$ the incident wave is transmitted around a corner of any angle without change of amplitude.

6. Discussion

For the special case $\gamma = \frac{1}{2}\pi$, or $\tau = 1$, we have from (5.6) and (5.2)

$$T = 1 \text{ for } \omega < f, \tag{6.1}$$

and $$T = |\tan^2 2\lambda| = \frac{4f^2\omega^2}{(f^2+\omega^2)^2} \quad \text{for} \quad \omega > f. \tag{6.2}$$

This is the problem considered by Buchwald (1968). He has calculated the transmission coefficient T only for the case $\omega < f$ and his result is identical to (6.1). He has not, however, given the result for $\omega > f$, but the corresponding calculation can be made and the result agrees with (6.2).

The result (5.6) for $\omega < f$ shows that the transmission coefficient is unity for all wedge angles. This may also be obtained from energy considerations, since for $\omega < f$ the energy contribution from the diffraction term is evanescent.

For $\omega > f$ it follows from (5.2) that

$$T = \left(\frac{\cosh 2\pi\lambda_0/\gamma - \cos \pi^2/\gamma}{\cosh 2\pi\lambda_0/\gamma + \cos \pi^2/\gamma}\right)^{\frac{1}{2}} \tanh\frac{\pi\lambda_0}{\gamma}, \tag{6.3}$$

where $\tanh \lambda_0 = f/\omega$. Hence

$$T \leqslant \left(\frac{\cosh 2\pi\lambda_0/\gamma + 1}{\cosh 2\pi\lambda_0/\gamma - 1}\right)^{\frac{1}{2}} \tanh\frac{\pi\lambda_0}{\gamma} = 1, \tag{6.4}$$

with equality if and only if $\gamma = \pi/(2n+1)$.

We have seen that for $\omega > f$ there is in general an outgoing cylindrical wave whose energy contribution at a large distance is finite, so that for the special wedge angles $\pi/(2n+1)$ there can be no cylindrical wave. Indeed, for these angles and $\omega > f$, it can be shown that

$$H(\nu) = -\frac{\cos(n+\frac{1}{2})(\nu-\lambda)\,\sin(\nu+\lambda)}{\sin(n+\frac{1}{2})(\nu+\lambda)\,\sin(\nu-\lambda)} \prod_{r=1}^{n} \frac{\sin(\nu+[2r\pi/(2n+1)]+\lambda)}{\sin(\nu+[2r\pi/(2n+1)]-\lambda)}. \tag{6.5}$$

In this special case, of course, (3.7) can be solved directly to give the above expression. It follows that

$$H(\nu) = H(2\pi+\nu), \tag{6.6}$$

and thus from (3.30), by applying the saddle-point method, there are no cylindrical waves. Further, in this case, it may be verified that

$$\zeta = \sum_{m=0}^{n} a_m \exp[-ikr\cos(\theta+2m\gamma-\lambda)] + \sum_{s=1}^{n} b_s \exp[-ikr\cos(\theta-2s\gamma+\lambda)], \tag{6.7}$$

where $\gamma = \pi/(2n+1)$ satisfies all the conditions provided

$$\left.\begin{aligned} a_n &= 1, \\ a_m &= \prod_{s=m}^{n-1} \frac{\sin\{(2s+1)\gamma-2\lambda\}}{\sin\{2(s+1)\gamma-2\lambda\}}\,\frac{\sin 2(s+1)\gamma}{\sin(2s+1)\gamma}, \quad (m = 0, 1, \ldots, (n-1)), \\ b_m &= a_m \frac{\sin 2m\gamma}{\sin(2m\gamma-2\lambda)}, \quad (m = 0, 1, \ldots, (n-1)). \end{aligned}\right\} \tag{6.8}$$

The a_m terms are Kelvin waves, the incident and transmitted waves being given by $m = n$, 0 respectively, and the b_m terms are Poincaré waves and attenuated in the fluid region.

From (6.8)

$$a_0 = B = \prod_{s=0}^{n-1} \frac{\sin\{(2s+1)\gamma-2\lambda\}}{\sin\{2(s+1)\gamma-2\lambda\}} = \prod_{s=0}^{n-1} \frac{\sin\{2(s+1)\gamma+2\lambda\}}{\sin\{2(s+1)\gamma-2\lambda\}}, \tag{6.9}$$

and this result can also be shown to follow from (6.5). It is easily seen that

$$T = |B| = 1. \tag{6.10}$$

In order to show the dependence of T on frequency and wedge angle for $\omega > f$, as given by (6.3), we have plotted T as a function of the wedge angle for four typical values of f/ω. Figure 1 shows the variation of T with wedge angle for $f/\omega = \frac{\sqrt{3}}{2}, \frac{2}{3}, \frac{2}{5}$ and $\frac{1}{5}$, where the first of these corresponds approximately to the semi-diurnal M_2 tide at the entrance to the North Sea. For a given angle not

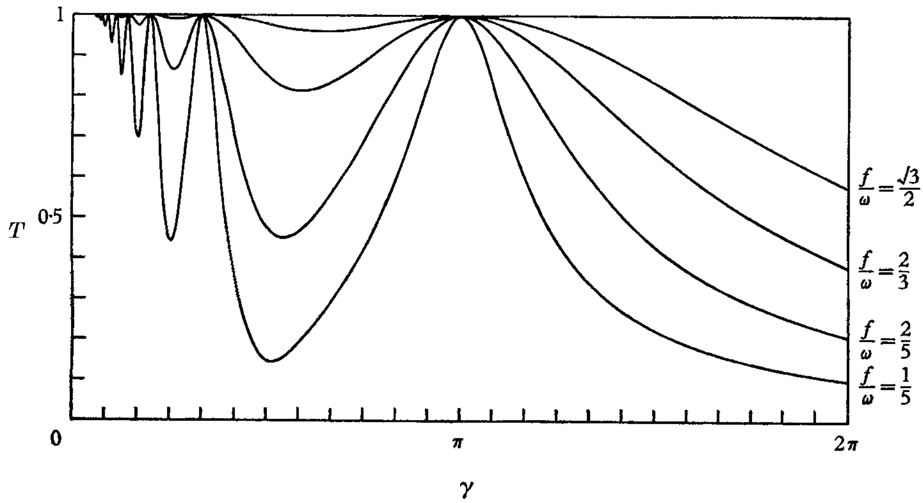

FIGURE 1. Graphs of T as a function of the wedge angle for $f/\omega = \frac{\sqrt{3}}{2}, \frac{2}{3}, \frac{2}{5}$ and $\frac{1}{5}$.

equal to $\pi/(2n+1)$ the amount of energy converted into Poincaré and cylindrical waves increases with increasing frequency. For small values of the wedge angle the graphs oscillate rapidly, due to the fact that for wedge angles $\pi/(2n+1)$ energy is transferred completely to the transmitted Kelvin wave, and thus for small angles the amount of energy transmitted is very critical, especially at high frequencies.

It is of interest to note that there is a marked minimum in the amplitude of the transmitted wave in the neighbourhood of $\frac{1}{2}\pi$, which is the case treated by Buchwald, and that in view of the complete reflexion at all frequencies for a 60° corner a Kelvin wave could be propagated round a sufficiently large equilateral triangular basin with very little attenuation.

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